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Viability for Semilinear Differential Inclusions via the Weak Sequential Tangency Condition

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Let X be a reflexive and separable Banach space, $A: D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t): X \to X$, $t \geq 0$, D a locally weakly sequentially closed subset in X, and $F: D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is weakly-weakly upper semi-continuous. The main result of the paper is:

THEOREM. *Under the general assumptions above a necessary and sufficient condition in order that for each $\xi \in D$ there exists at least one mild solution u of $\frac{du}{dt}(t) \in Au(t) + F(u(t))$ satisfying $u(0) = \xi$ is the so-called "weak sequential tangency condition" below.*

($\mathscr{WSTC}$) *For each $\xi \in D$ there exists $y \in F(\xi)$ and two sequences $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ and $(p_n)_{n \in \mathbb{N}}$ in X such that $t_n \to 0$, $p_n \to 0$ and satisfying $S(t_n)\xi + t_n(y + p_n) \in D$.* © 2001 Academic Press

1. INTRODUCTION

The starting point of this paper lies in [4] where the authors proved a necessary and sufficient condition in order that a given subset of a Banach space X be a viable domain for a semilinear differential inclusion. Namely, let X be a Banach space, $A: D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t): X \to X$, $t \geq 0$, D a nonempty subset in X, and $F: D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping and let us consider the semilinear differential inclusion

$$\frac{du}{dt}(t) \in Au(t) + F(u(t)), \qquad t \geq 0. \qquad (\mathscr{DI})$$

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We say that D is a *viable domain* for $(\mathscr{DI})$ if for each $\xi \in D$ there exists at least one mild solution u: $[0, T] \to D$ of $(\mathscr{DI})$ satisfying the initial condition

$$u(0) = \xi. \tag{$\mathscr{IC}$}$$

We recall that the function u: $[0, T] \to D$ is a *mild solution* of $(\mathscr{DI})$ and $(\mathscr{IC})$ if there exists $f \in L^1(0, T; X)$, with $f(t) \in F(u(t))$ a.e. for $t \in (0, T)$ and such that

$$u(t) = S(t)\xi + \int_0^t S(t-s)f(s)\,ds$$

for $t \in [0, T]$.

The main result in [4] is

THEOREM 1.1. *Let X be a reflexive and separable Banach space, A: $D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t)$: $X \to X$, $t \geq 0$, D a locally weakly closed subset in X, and F: $D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is locally weakly-weakly upper semi-continuous. Then a necessary and sufficient condition in order that D be a viable domain for $(\mathscr{DI})$ is the so-called "bounded w-tangency condition" below.*

$(\mathscr{BWTC})$ *There exists a locally bounded function $\mathscr{M}$: $D \to \mathbb{R}_+^*$ such that for each $\xi \in D$ there exists $y \in F(\xi)$ such that for each $\delta > 0$ and each weak neighborhood V of 0 there exist $t \in (0, \delta]$ and $p \in V$ with $\|p\| \leq \mathscr{M}(\xi)$ and satisfying*

$$S(t)\xi + t(y + p) \in D.$$

Our aim here is to show that the boundedness assumption above, i.e., the existence of the locally bounded function $\mathscr{M}$ which, in view of Theorem 1.1, is necessary for viability whenever we use a tangency concept defined in the terms of generalized sequences, can be discarded if we consider the sequential counterpart of that weak tangency concept.

There is a rich literature on the viability problem starting with the pioneering work of Nagumo [11] who considered the finite dimensional case, $A = 0$, and F single-valued and continuous on the closed subset D. The multivalued finite-dimensional case has been considered by Haddad [9] (again with $A = 0$) who showed that, whenever F is a nonempty, compact, convex valued upper semi-continuous (u.s.c.) mapping, a necessary and sufficient condition in order that the locally closed set D be a viable domain for $(\mathscr{DI})$ is the tangency condition:

For each $\xi \in D$ there exist $y \in F(\xi)$, a sequence $(h_n)_{n \in \mathbb{N}}$ decreasing to 0, *and a sequence $(p_n)_{n \in \mathbb{N}}$ convergent to 0 satisfying*

$$\xi + h_n(y + p_n) \in D \tag{1.1}$$

for each $n \in \mathbb{N}$.

For results, references, and applications we refer to Aubin [1] for the finite dimensional setting and to Aubin and Cellina [2] and Motreanu and Pavel [10] for the infinite dimensional one. Brief reviews of the main contributions in this area can be found in Cârjă and Vrabie [4–6].

However, in order to help the reader catch the significance of our results here, we shall recall some previous important contributions in the infinite dimensional case.

We begin by recalling the pioneering work of Gautier [8] who assumed that D is weakly closed, $A = 0$, and F is weakly-weakly upper semi-continuous. Accordingly he has been led to use a weak tangential sufficient condition of the form:

For each $\xi \in D$ *there exist* $y \in F(\xi)$, *a sequence* $(h_n)_{n \in \mathbb{N}}$ *decreasing to* 0, *and a sequence* $(p_n)_{n \in \mathbb{N}}$ *weakly convergent to* 0 *with* $\|p_n + y\| \le 2\|y\|$ *and satisfying* (1.1) *for each* $n \in \mathbb{N}$.

As far as we know, the true semilinear and multivalued case, i.e., A unbounded and F multivalued, was considered first by Pavel and Vrabie [13, 14] at the end of the seventies. For subsequent developments see Aubin [1], Shi Shuzhong [17], Cârjă and Vrabie [4], and the references therein. We recall that Pavel and Vrabie [13, 14] assumed that D is locally closed (in fact they assume a strictly weaker condition on D), $F\colon D \to 2^X$ is a nonempty, closed and convex valued mapping which is locally bounded and whose graph is strongly $\times$ weakly sequential closed, and A generates a compact C_0-semigroup. Using this general setting they proved that a sufficient condition in order that D be a viable domain for $(\mathscr{D}\mathscr{T})$ is the tangency condition:

For each $\xi \in D$ *and each* $y \in F(\xi)$

$$\lim_{h \downarrow 0} \frac{1}{h} d(S(h)\xi + hy, D) = 0. \tag{1.2}$$

We emphasize that this sort of tangency condition which may hold in points which do not belong to the domain of the right-hand side (we recall that A is only densely defined) has been formulated for the first time by Pavel [12] in the case in which F is single-valued and continuous. We note that, whenever $\xi \in D \cap D(A)$, (1.2) is equivalent to

$$\lim_{h \downarrow 0} \frac{1}{h} d(\xi + h(A\xi + y), D) = 0 \tag{1.3}$$

which is nothing else than (1.1) with F replaced by $A + F$. However, there exist situations in which D is not included in $D(A)$, or even worse, when $D \cap D(A) = \emptyset$. Think of the case when D is the trajectory of a nowhere differentiable mild solution of $(\mathscr{D}\mathscr{T})$. In all these cases (1.3) is meaningless and we can use only (1.2).

In order to help the reader make an accurate comparison between our main result and those mentioned above, it would be useful to observe that (1.2) may be equivalently expressed as:

For each $\xi \in D$, *each* $y \in F(\xi)$, *and each sequence* $(t_n)_{n \in \mathbb{N}}$ *decreasing to* 0, *there exists a sequence* $(p_n)_{n \in \mathbb{N}}$ *strongly convergent to* 0 *such that*

$$S(t_n)\xi + t_n(y + p_n) \in D \tag{1.4}$$

for each $n \in \mathbb{N}$.

As we can easily see, the general assumptions on D and F are natural and less restrictive, but the compactness of the semigroup precludes the applicability of the abstract result of Pavel and Vrabie [13, 14] to other than parabolic or, of course, finite-dimensional problems. More than this, the tangency condition is too strong, at least when compared with its finite-dimensional counterpart (1.1).

Shi Shuzhong [17] considers the case in which D is compact, F is a nonempty, convex, and compact valued mapping which is strongly-strongly u.s.c., and A generates a compact, differentiable C_0-semigroup. Under these circumstances, he proved that a necessary and sufficient condition in order that D be a viable domain for $(\mathscr{D}\mathscr{T})$ is the following tangency condition:

For each $\xi \in D$ *there exist* $y \in F(\xi)$, *a sequence* $(t_n)_{n \in \mathbb{N}}$ *decreasing to* 0, *and a sequence* $(p_n)_{n \in \mathbb{N}}$ *strongly convergent to* 0 *such that* (1.4) *holds for each* $n \in \mathbb{N}$.

Clearly, in this case the general assumptions on D and F are significantly stronger than those in Pavel and Vrabie [13, 14]. We note that for instance, in the infinite-dimensional setting, the compactness of $F(\xi)$ for each $\xi \in D$ is not satisfied if F is a superposition operator which is not single-valued. On the other hand, this "weakness" of the general setting of Shi Shuzhong [17] is well counterbalanced by the tangency condition which is quite close to its finite-dimensional counterpart.

In the present paper we assume that D is locally weakly sequentially closed in the sense that for each $\xi \in D$ there exists $r > 0$ such that $D \cap B(\xi, r)$ is weakly sequentially closed in X, where, as usual, $B(\xi, r)$ denotes the closed ball with center ξ and radius r. Concerning F we assume that it is a nonempty, closed, convex valued multifunction which is locally weakly-weakly u.s.c.; i.e., for each $\xi \in D$ there exists $r > 0$ such that the restriction of F to $D \cap B(\xi, r)$ is weakly-weakly upper semicontinuous in the sense that for each $u \in D \cap B(\xi, r)$ and each neighborhood V of $F(u)$ in the weak topology there exists a neighborhood W of u, also in the weak topology, such that $F(v) \subset V$ for each $v \in W \cap D \cap B(\xi, r)$. At first glance, the weak-weak upper semicontinuity condition on F seems to be very restrictive, but it is not as we can see from Examples

2.1 and 2.2 in Cârjă and Vrabie [4]. Moreover, we emphasize that our Theorem 2.1 extends considerably the main results in Gautier [8], Haddad [9], Shi Shuzhong [17] and, in the specific case in which D is weakly locally closed, those in Pavel and Vrabie [13, 14] (where the condition on D is strictly less restrictive than here).

The paper is divided into four sections. The second one is mainly concerned with the statement of the main viability theorem. The proof of our result is contained in the third section, while in the fourth one we include an application concerning the existence of monotone trajectories.

2. STATEMENT OF THE MAIN RESULT

We assume familiarity with the basic concepts and results concerning C_0-semigroups and multivalued mappings and we refer to Pazy [15], Aubin and Cellina [2], and Vrabie [18] for details.

We introduce first the tangency condition we are going to use in the sequel. We begin with the tangency concept. We say that $y \in X$ is *weakly sequentially A-tangent* to D at $\xi \in D$ if there exist a sequence $(t_n)_{n \in \mathbb{N}}$ decreasing to 0 and a sequence $(p_n)_{n \in \mathbb{N}}$ weakly convergent to 0 such that

$$S(t_n)\xi + t_n(y + p_n) \in D$$

for each $n \in \mathbb{N}$.

The set of all *weakly sequentially A-tangent* elements to D at $\xi \in D$ is denoted by $\mathscr{WST}_D^A(\xi)$.

We say that the set D satisfies the *weak-sequential-tangency condition* ($\mathscr{WSTC}$) with respect to ($\mathscr{DT}$) if

$$F(\xi) \cap \mathscr{WST}_D^A(\xi) \neq \emptyset$$

for each $\xi \in D$.

PROPOSITION 2.1. *An element $y \in X$ is weakly sequentially A-tangent to D at ξ if and only if there exists a sequence $(t_n)_{n \in \mathbb{N}}$ decreasing to 0 and a sequence $(p_n)_{n \in \mathbb{N}}$ weakly convergent to 0 such that*

$$S(t_n)\xi + \int_0^{t_n} S(t_n - s)y\,ds + t_n p_n \in D$$

for each $n \in \mathbb{N}$.

Since the proof of Proposition 2.1 follows exactly the same lines as those in the proof of Proposition 2.1 in Cârjă and Vrabie [4] we do not give details.

Remark 2.1. If X is finite dimensional and $A = 0$, then the weak-sequential-tangency condition $(\mathscr{WSTC})$ reduces to the classical one used by Haddad [9], i.e.,

$$F(\xi) \cap \mathscr{T}_D(\xi) \neq \emptyset \qquad (\mathscr{CTC})$$

for each $\xi \in D$. Here $\mathscr{T}_D(\xi) = \mathscr{WST}_D^0(\xi)$ is the tangent cone of Bouligand [3] and Severi [16], which can be equivalently defined as

$$\mathscr{T}_D(\xi) = \left\{ y \in X;\ \liminf_{t \downarrow 0} \frac{1}{t} d(\xi + ty, D) = 0 \right\}.$$

Remark 2.2. Again in the case in which X is finite dimensional, $A = 0$, D locally closed, and F: $D \to 2^X$ u.s.c. with nonempty, compact, convex values, the classical tangency condition $(\mathscr{CTC})$ is necessary and sufficient in order that D be a viable domain for $(\mathscr{DI})$. See for instance Haddad [9] or Aubin and Cellina [2].

We may now proceed to the statement of the main result of this paper.

THEOREM 2.1. *Let X be a reflexive and separable Banach space, D a nonempty, weakly locally closed subset in X, and F: $D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is locally weakly-weakly u.s.c. Let A: $D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)$: $X \to X$, $t \geq 0$. Then a necessary and sufficient condition in order that D be a viable domain for $(\mathscr{DI})$ is the $(\mathscr{WSTC})$.*

It is important to remark that the $(\mathscr{WSTC})$ is a necessary condition in order that D be a viable domain for $(\mathscr{DI})$ in a more general setting. Namely, we have:

THEOREM 2.2. *Let X be a reflexive Banach space, D a nonempty, locally closed subset in X, and F: $D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is strongly weakly u.s.c. and locally bounded. Let A: $D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)$: $X \to X$, $t \geq 0$. Then a necessary condition in order that D be a viable domain for $(\mathscr{DI})$ is the $(\mathscr{WSTC})$.*

Taking into account that if D is strongly locally compact and F: $D \to 2^X$ is strongly weakly u.s.c. it is weakly-weakly u.s.c., from Theorem 2.1, we easily deduce:

COROLLARY 2.1. *Let X be a reflexive and separable Banach space, D a nonempty, strongly locally compact subset in X, F: $D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is strongly weakly u.s.c. and A: $D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t)$: $X \to X$, $t \geq 0$.*

Then a necessary and sufficient condition in order that D be a viable domain for $(\mathscr{DI})$ *is* $(\mathscr{WSTC})$.

Remark 2.3. We emphasize that Corollary 2.1 represents a considerable extension of the sufficiency part of Shi Shuzhong's main result in [17], where, in addition to our general conditions, it is also assumed that A generates a compact, differentiable C_0-semigroup and F is compact valued. It should be noted that Shi Shuzhong [17] does not assume explicitly that X is separable, but this follows from the fact that the C_0-semigroup generated by A is compact. See for instance Proposition 4.1 in Cârjă and Vrabie [6]

Remark 2.4. The compactness condition on D imposed in Corollary 2.1 is rather natural if, for instance, A comes from a parabolic problem, but it is not appropriate if A is a "hyperbolic" operator. However, in this last case, if F is the superposition operator associated to an u.s.c. mapping G: $\mathbb{R} \to 2^{\mathbb{R}}$ then (surprisingly!) F is a fortiori locally weakly-weakly u.s.c. See Example 2.2 in Cârjă and Vrabie [4]. So the weak-weak upper semi-continuity condition on F is not so restrictive as it seems at first glance.

Concerning the existence of saturated, i.e., noncontinuable mild solutions, from Theorem 2.1, using very similar arguments as in Theorem 3.2.1 of Vrabie [18, p. 92] we deduce:

THEOREM 2.3. *Let X be a reflexive and separable Banach space, D a nonempty, weakly locally closed subset in X, and F: $D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is locally weakly-weakly u.s.c. Let A: $D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)$: $X \to X$, $t \geq 0$. Then a necessary and sufficient condition in order that for each $\xi \in D$ there exists at least one saturated mild solution of* $(\mathscr{DI})$ *satisfying* $(\mathscr{TC})$ *is the* $(\mathscr{WSTC})$.

The next class of multivalued mappings has been introduced in Vrabie [18, Definition 3.2.1, p. 95] in order to handle in a unitary frame both the sign condition and the linear growth condition on F usually imposed to obtain global existence results. First we recall that for each $\xi, \eta \in X$

$$[\xi, \eta]_+ = \lim_{h \downarrow 0} \frac{1}{h}(\|\xi + h\eta\| - \|\xi\|).$$

We say that a mapping F: $D \to 2^X$ is *positively sublinear* if there exist $a > 0$, $b \in \mathbb{R}$, and $k > 0$ such that

$$\sup\{\|y\|;\ y \in F(\xi)\} \leq a\|\xi\| + b$$

for each $\xi \in X_+^k(F)$, where

$$X_+^k(F) := \{\xi \in D;\ \sup\{[\xi, y]_+;\ y \in F(\xi)\} > 0,\ \|\xi\| > k\}.$$

As concerns the existence of global mild solutions of $(\mathscr{D}\mathscr{I})$, i.e., mild solutions defined on $\mathbb{R}_+$, reasoning as in Theorem 3.2.3 of Vrabie [18, p. 96], we get:

THEOREM 2.4. *Let X be a reflexive and separable Banach space, D a nonempty, weakly sequentially closed subset in X, and $F\colon D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is locally weakly-weakly u.s.c. and positive sublinear. Let $A\colon D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)\colon X \to X$, $t \geq 0$. Then a necessary and sufficient condition in order that for each $\xi \in D$ there exists at least one global mild solution of $(\mathscr{D}\mathscr{T})$ satisfying $(\mathscr{T}\mathscr{C})$ is the $(\mathscr{W}\mathscr{S}\mathscr{T}\mathscr{C})$.*

Noticing that whenever D is weakly compact each weakly compact valued mapping $F\colon D \to 2^X$, which is weakly-weakly u.s.c., is a fortiori globally bounded, from Theorem 2.4 it readily follows:

COROLLARY 2.2. *Let X be a reflexive and separable Banach space, D a nonempty, weakly compact subset in X, and $F\colon D \to 2^X$ a nonempty, closed, convex, and bounded valued mapping which is weakly-weakly u.s.c. Let $A\colon D(A) \subset X \to X$ be the generator of a C_0-semigroup $S(t)\colon X \to X$, $t \geq 0$. Then a necessary and sufficient condition in order that for each $\xi \in D$ there exists at least one global mild solution of $(\mathscr{D}\mathscr{T})$ satisfying $(\mathscr{T}\mathscr{C})$ is the $(\mathscr{W}\mathscr{S}\mathscr{T}\mathscr{C})$.*

3. PROOF OF THEOREM 2.1

The necessity follows from Theorem 2.2 in Cârjă and Vrabie [4]. The proof of sufficiency consists in showing that the $(\mathscr{W}\mathscr{S}\mathscr{T}\mathscr{C})$ along with Zorn's Lemma implies that for each $\xi \in D$ there exists at least one sequence of "approximate solutions" of $(\mathscr{D}\mathscr{T})$, defined on the same interval, $u_n\colon [0, T] \to X$, satisfying $(\mathscr{T}\mathscr{C})$ for each $n \in \mathbb{N}^*$ and such that $(u_n)_{n \in \mathbb{N}}$ converges in some sense to a mild solution of $(\mathscr{D}\mathscr{T})$ satisfying $(\mathscr{T}\mathscr{C})$.

The next lemma represents an existence result concerning "approximate" solutions of $(\mathscr{D}\mathscr{T})$ satisfying $(\mathscr{T}\mathscr{C})$.

LEMMA 3.1. *Let X be a real Banach space, D a nonempty, strongly locally closed subset in X, $F\colon D \to 2^X$ a nonempty valued mapping which is locally bounded, and $A\colon D(A) \subset X \to X$ the generator of a C_0-semigroup $S(t)\colon X \to X$, $t \geq 0$. If D satisfies the $(\mathscr{W}\mathscr{S}\mathscr{T}\mathscr{C})$ with respect to $(\mathscr{D}\mathscr{T})$ then for each $\xi \in D$ there exist $r > 0$, $T > 0$, $K > 0$ such that $D \cap B(\xi, r)$ is strongly closed and for each weak neighborhood V of the origin and each $n \in \mathbb{N}^*$ there* *exist five measurable functions* $f\colon [0,T] \to X$, $g\colon [0,T] \to X$, $\alpha\colon [0,T] \to [0,T]$, $\beta\colon \{(t,s);\ 0 \le s < t \le T\} \to [0,T]$ *and* $u\colon [0,T] \to X$ *satisfying*

(i) $s - \frac{1}{n} \le \alpha(s) \le s$, $u(\alpha(s)) \in D \cap B(\xi, r)$ *and* $f(s) \in F(u(\alpha(s)))$, *a.e. for* $s \in [0,T]$

(ii) $\|f(s)\| \le K$ *a.e. for* $s \in [0,T]$

(iii) $u(T) \in D \cap B(\xi, r)$

(iv) $g(s) \in V$, $\|g(s)\| \le 1/\sqrt{s}$ *a.e. for* $s \in [0,T]$, $\beta(t,s) \le t$ *for* $0 \le s < t \le T$, *and* $t \mapsto \beta(t,s)$ *is nonexpansive on* $(s,T]$ *and*

$$u(t) = S(t)\xi + \int_0^t S(t-s)f(s)\,ds + \int_0^t S(\beta(t,s))g(s)\,ds \tag{3.1}$$

for each $t \in [0,T]$.

Proof. Let $\xi \in D$ be arbitrary and choose $r > 0$, $T > 0$, and $K > 0$ such that $D \cap B(\xi, r)$ is strongly closed and

$$\|y\| \le K \tag{3.2}$$

for each $x \in D \cap B(\xi, r)$ and $y \in F(x)$, and

$$\sup_{0 \le t \le T} \|S(t)\xi - \xi\| + Me^{\omega T}\big(TK + \sqrt{T}\big) \le r, \tag{3.3}$$

where $M > 0$ and $\omega \ge 0$ are such that

$$\|S(t)\| \le Me^{\omega t}$$

for each $t \ge 0$. This is always possible since D is strongly locally closed, F is locally bounded, and $S(t)\colon X \to X$, $t \ge 0$, is a C_0-semigroup.

Let $n \in \mathbb{N}^*$ and let V be a weak neighborhood of the origin.

We start by showing how to define the functions f, g, α, β, and u on a sufficiently small interval $[0, \delta]$ and then we will show how to extend them to the whole interval $[0,T]$.

Because of ($\mathscr{WSTC}$), in view of Proposition 2.1, there exist $y \in F(\xi)$, a sequence $(\delta_k)_{k \in \mathbb{N}}$ in $\mathbb{R}_+$, and a sequence $(p_k)_{k \in \mathbb{N}}$ satisfying $\lim_k \delta_k = 0$, $\lim_k p_k = 0$ weakly in X and

$$S(\delta_k)\xi + \int_0^{\delta_k} S(\delta_k - s)y\,ds + \delta_k p_k \in D.$$

Since $(p_k)_{k \in \mathbb{N}}$ is weakly convergent to 0 it is a fortiori bounded and therefore, there exists $m \in \mathbb{N}$ such that $\sqrt{\delta_m}\,\|p_m\| \le 1$ and $p_m \in V$. Take $\delta = \delta_m$, $p = p_m$, and define $u\colon [0, \delta] \to X$ as

$$u(t) = S(t)\xi + \int_0^t S(t-s)y\,ds + tp \tag{3.4}$$

for each $t \in [0, \delta]$. In view of (3.2), (3.3) we have:

(j) $y \in F(u(0))$

(jj) $\|y\| \leq K$

(jjj) $u(\delta) \in D \cap B(\xi, r)$

(jv) $p \in V$ and $\|p\| \leq 1/\sqrt{\delta}$ (so $\|p\| \leq 1/\sqrt{t}$ for each $t \in (0, \delta)$).

Setting $f(s) = y$, $g(s) = p$, $\alpha(s) = 0$, and $\beta(t, s) = 0$ for $s \in [0, \delta]$ and $0 \leq s < t \leq \delta$, from (j)–(jv) and (3.4), we can easily see that (f, g, α, β, u) satisfies (i)–(iv) and (3.1) with T substituted by δ.

Next, we are going to show that for each $n \in \mathbb{N}^*$ and each weak neighborhood V of the origin there exists at least one 5-tuple (f, g, α, β, u) whose domain is denoted for simplicity by $D(T)$, where, for $a \in \mathbb{R}_+$

$$D(a) = [0, a] \times [0, a] \times [0, a] \times \{(t, s);\ 0 \leq s < t \leq a\} \times [0, a],$$

satisfying (i)–(iv) and (3.1). To this aim we shall use Zorn's Lemma as follows. Let $\mathscr{U}$ be the set of all 5-tuples (f, g, α, β, u) defined on $D(a)$ with $a \leq T$ and satisfying (i)–(iv) and (3.1) on $[0, a]$. This set is clearly nonempty because (f, g, α, β, u) defined as above belongs to $\mathscr{U}$. On $\mathscr{U}$ we introduce a partial order as follows. We say that $(f_u, g_u, \alpha_u, \beta_u, u)$ defined on $D(a)$ and $(f_v, g_v, \alpha_v, \beta_v, v)$ defined on $D(b)$ satisfy

$$(f_u, g_u, \alpha_u, \beta_u, u) \leq (f_v, g_v, \alpha_v, \beta_v, v)$$

if $a \leq b$, $f_u(s) = f_v(s)$, $g_u(s) = g_v(s)$, $\alpha_u(s) = \alpha_v(s)$ a.e. for $s \in [0, a]$ and $\beta_u(t, s) = \beta_v(t, s)$ for $0 \leq s < t \leq a$.

Let $\mathscr{L}$ be a chain in $\mathscr{U}$,

$$\mathscr{L} = \{(f_i, g_i, \alpha_i, \beta_i, u_i) \colon D(a_i) \to X \times X \times [0, a_i] \times [0, a_i] \times X;\ i \in \mathbb{I}\}.$$

We define a majorant of $\mathscr{L}$ as follows. First set

$$a^* \coloneqq \sup\{a_i; i \in \mathbb{I}\}.$$

If $a^* = a_i$ for some $i \in \mathbb{I}$, $(f_i, g_i, \alpha_i, \beta_i, u_i)$ is clearly a majorant for $\mathscr{L}$. If $a_i < a^*$ for each $i \in \mathbb{I}$, we may assume with no loss of generality that $\mathbb{I} = \mathbb{N}$. We define

$$f(s) = f_i(s), \qquad g(s) = g_i(s), \qquad \alpha(s) = \alpha_i(s)$$

for $i \in \mathbb{N}$ and a.e. for $s \in [0, a_i]$ and

$$\beta(t, s) = \begin{cases} \beta_i(t, s) & \text{if } 0 \leq s < t \leq a_i < a^* \\ \lim\limits_i \beta_i(a_i, s) & \text{if } 0 \leq s < a_i < t = a^*. \end{cases}$$

Now let us observe that (f, g, α, β, u), where f, g, α, and β are defined as above while u is given by (3.1), satisfies (i), (ii), and (iv). Moreover, since for each $s \in [0, a^*)$, $t \mapsto \beta(t, s)$ is nonexpansive on $(s, a^*]$ and $f, g \in L^\infty(0, a^*; X)$, a simple argument involving (3.1) and the Lebesgue Dominated Convergence Theorem show that u is continuous on $[0, a^*]$. Consequently there exists

$$\lim_i u_i(a_i) = u^* = u(a^*) \tag{3.5}$$

strongly in X. Then, since by (iii), $u_i(a_i) \in D \cap B(\xi, r)$ for each $i \in \mathbb{N}$ and $D \cap B(\xi, r)$ is strongly closed in X, by (3.5), we easily conclude that u satisfies (iii) too. In addition

$$(f_i, g_i, \alpha_i, \beta_i, u_i) \le (f, g, \alpha, \beta, u)$$

for each $i \in \mathbb{N}$ and thus $\mathscr{U}$ endowed with the partial order $\le$ satisfies the hypotheses of Zorn's Lemma. Consequently, there exists at least one maximal element (f, g, α, β, u) in $\mathscr{U}$ whose domain is $D(a)$.

Let us show that $a = T$. To this aim let us assume by contradiction that $a < T$ and let $\xi_a := u(a)$ which belongs to $D \cap B(\xi, r)$. From (i), (ii), (iii), (iv), (3.1), (3.2), (3.3), and (3.4) we get

$$\begin{aligned}
\|\xi_a - \xi\| &\le \|S(a)\xi - \xi\| + \int_0^a \|S(a-s)f(s)\|\,ds \\
&\quad + \int_0^a \|S(\beta(a,s))g(s)\|\,ds \\
&\le \sup_{0 \le t \le a} \|S(t)\xi - \xi\| + Me^{\omega a}\left(aK + \sqrt{a}\right) < r.
\end{aligned}$$

Using once again $(\mathscr{WSTC})$ and Proposition 2.1 combined with the inequality above we infer that there exist $y_a \in F(\xi_a)$, $\delta_a \in (0, \frac{1}{n}]$ with $a + \delta_a \le T$ and $p_a \in V$ satisfying $\|p_a\| \le 1/\sqrt{\delta_a}$, such that

$$S(\delta_a)\xi_a + \int_0^{\delta_a} S(\delta_a - s)y_a\,ds + \delta_a p_a \in D \cap B(\xi, r). \tag{3.6}$$

We define $\tilde{f}$, $\tilde{g}$, and $\tilde{\alpha}$ on $[0, a + \delta_a]$ by f, g and respectively α on $[0, a]$ and by y_a, p_a and respectively a on $[a, a + \delta_a]$. Furthermore, we define $\tilde{\beta}$ and $\bar{u}$ by

$$\tilde{\beta}(t,s) = \begin{cases} \beta(t,s) & \text{if } 0 \le s < t \le a \\ t - a + \beta(a,s) & \text{if } 0 \le s < a < t \le a + \delta_a \\ 0 & \text{if } a \le s < t \le a + \delta_a \end{cases}$$

and respectively by

$$\tilde{u}(t) = S(t)\xi + \int_0^t S(t-s)\tilde{f}(s)\,ds + \int_0^t S\big(\tilde{\beta}(t,s)\big)\tilde{g}(s)\,ds$$

for $t \in [0, a+\delta_a]$.

To get a contradiction it suffices to show that $(\tilde{f}, \tilde{g}, \tilde{\alpha}, \tilde{\beta}, \tilde{u}) \in \mathscr{U}$. Clearly $\tilde{f}$, $\tilde{g}$, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{u}$ satisfy (i), (ii), (iv), and (3.1). To show (iii), let us observe first that

$$\tilde{u}(t) = S(t-a)\xi_a + \int_a^t S(t-s)y_a\,ds + (t-a)p_a$$

for $t \in [a, a+\delta_a]$ and therefore $\tilde{u}(a+\delta_a) \in D$. Furthermore, from (3.6), (i), (ii), (iv) and (3.2), (3.3), and (3.4), we get

$$\begin{aligned}\|\tilde{u}(t) - \xi\| &\le \|S(t)\xi - \xi\| + \int_0^t \|S(t-s)\tilde{f}(s)\|\,ds \\ &\quad + \int_0^t \|S\big(\tilde{\beta}(t,s)\big)\tilde{g}(s)\|\,ds \\ &\le \sup_{0\le t\le T} \|S(t)\xi - \xi\| + Me^{\omega T}\big(TK + \sqrt{T}\big) \le r\end{aligned}$$

for each $t \in [0, a+\delta_a]$ and therefore $\tilde{u}(a+\delta_a) \in B(\xi, r)$. Hence (iii) is also satisfied and consequently $(\tilde{f}, \tilde{g}, \tilde{\alpha}, \tilde{\beta}, \tilde{u}) \in \mathscr{U}$. Clearly

$$(f, g, \alpha, \beta, u) \le \big(\tilde{f}, \tilde{g}, \tilde{\alpha}, \tilde{\beta}, \tilde{u}\big) \quad \text{and} \quad (f, g, \alpha, \beta, u) \ne \big(\tilde{f}, \tilde{g}, \tilde{\alpha}, \tilde{\beta}, \tilde{u}\big)$$

while (f, g, α, β, u) is maximal in $\mathscr{U}$. This contradiction can be eliminated only if each maximal element in $\mathscr{U}$ is defined on $D(T)$ and this completes the proof of Lemma 3.1. ∎

Let $\xi \in D$, $n \in \mathbb{N}^*$, and V a weak neighborhood of the origin. A 5-tuple (f, g, α, β, u) satisfying (i)–(iv) and (3.1) is called an *(n, V)-approximate solution* of $(\mathscr{D}\mathscr{T})$ and $(\mathscr{T}\mathscr{C})$.

We are now prepared to complete the proof of Theorem 2.1.

Proof. Sufficiency. First, let

$$\rho = \max\{1, \|\xi\| + r\}$$

and let $\mathscr{V} = \{V_n;\, n \in \mathbb{N}^*\}$ be a countable fundamental system of neighborhoods of the origin in the weak topology of $B(0, \rho)$. We note that the existence of such a system is ensured by the fact that X is reflexive and separable.

Let $n \in \mathbb{N}^*$ and let us fix one (n, V_n)-approximate solution $(f_n, g_n, \alpha_n, \beta_n, u_n)$ of $(\mathscr{DT})$ and $(\mathscr{TC})$ defined on $[0, T]$.

Since, by (iv), $g_n(t) \in V_n$ for each $n \in \mathbb{N}^*$ and a.e. for $t \in [0, T]$ and $\beta_n(t, s) \le t$ for $t \in [0, T]$, we have

$$\lim_n \int_0^t S(\beta_n(t, s)) g_n(s)\, ds = 0$$

in the weak topology of X uniformly for $t \in [0, T]$. Moreover, as X is reflexive, from (ii) we may assume with no loss of generality (by extracting a subsequence if necessary) that there exists $f \in L^2(0, T; X)$ such that

$$\lim_n f_n = f$$

weakly in $L^2(0, T; X)$. As a consequence, from (3.1), we infer that there exists u: $[0, T] \to X$ such that

$$\lim_n u_n(t) = u(t)$$

in the weak topology of X uniformly for $t \in [0, T]$. Also from (3.1) and the last three relations, we easily conclude that

$$u(t) = S(t)\xi + \int_0^t S(t - s) f(s)\, ds$$

for each $t \in [0, T]$. Recalling that, by (i), we have $s - \frac{1}{n} \le \alpha_n(s) \le s$ for each $n \in \mathbb{N}^*$ and a.e. for $s \in [0, T]$, from the remarks above we obtain

$$\lim_n u_n(\alpha_n(s)) = u(s)$$

weakly in X a.e. for $s \in [0, T]$. Furthermore, again by (i), $u_n(\alpha_n(s)) \in D \cap B(\xi, r)$ a.e. for $s \in [0, T]$ and $D \cap B(\xi, r)$ is weakly sequentially closed, we have $u(s) \in D$ a.e. for $s \in [0, T]$. Since u is continuous and $D \cap B(\xi, r)$ is strongly closed (being weakly sequentially closed), we have in fact $u(s) \in D \cap B(\xi, r)$ for each $s \in [0, T]$. Finally, since F is weakly-weakly u.s.c., by virtue of Theorem 3.1.2 of Vrabie [18 p. 88], we conclude that $f(s) \in F(u(s))$ a.e. for $s \in [0, T]$, and thus u is a mild solution of $(\mathscr{DT})$ and $(\mathscr{TC})$. ∎

4. MONOTONE TRAJECTORIES

Let M be a nonempty subset of D and $\preceq$ be a preorder on M, i.e., a reflexive and transitive binary relation on M. It is convenient to identify the preorder $\preceq$ on M with the multifunction P: $M \to 2^M$ defined by

$$P(\xi) = \{\eta \in M;\ \xi \preceq \eta\}$$

for all $\xi \in M$.

We say that the preorder $P: M \to 2^M$ is *admissible* with respect to $(\mathscr{DI})$ if for every $\xi \in M$ there exists a mild solution $u: [0,T] \to M$ to $(\mathscr{DI})$ and $(\mathscr{IC})$ such that for every $s \in [0,T]$ and for every $t \in [s,T]$, $u(t) \in P(u(s))$.

Our main goal in the sequel is to characterize the admissibility of a given preorder P with respect to the differential inclusion $(\mathscr{DI})$. The next result is a sequential version of Theorem 3.1 in Chiş-Şter [7].

THEOREM 4.1. *Under the hypotheses of Theorem* 2.1 *assume that M is weakly sequentially closed in D and the graph of P is weakly $\times$ weakly sequentially closed in $M \times M$. Then a necessary and sufficient condition in order that P is admissible with respect to $(\mathscr{DI})$ is the tangency condition*:

$(\mathscr{T})$ *For each $\xi \in M$ we have $F(\xi) \cap \mathscr{WST}^A_{P(\xi)}(\xi) \neq \emptyset$.*

Proof. The necessity being obvious, we focus our attention on the proof of the sufficiency. To this aim, let us observe that, in the general setting of Theorem 2.1, the admissibility of P with respect to $(\mathscr{DI})$ is equivalent to the viability of $P(\xi)$ with respect to $(\mathscr{DI})$. See Proposition 3.1 in Chiş-Şter [7]. On the other hand, since for each $\xi \in M$ and $\eta \in P(\xi)$, $P(\eta) \subset P(\xi)$, we easily conclude that the tangency condition $(\mathscr{T})$ implies:

For each $\xi \in M$ and each $\eta \in P(\xi)$ we have $F(\eta) \cap \mathscr{WST}^A_{P(\xi)}(\xi) \neq \emptyset$

which clearly shows that for each $\xi \in M$, $P(\xi)$ satisfies $(\mathscr{WSTC})$. By Theorem 2.1 we then conclude that for each $\xi \in M$, $P(\xi)$ is a viable domain for $(\mathscr{DI})$. An appeal to Proposition 3.1 in Chiş-Şter [7] completes the proof. ▮

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